# Rigorous Evanescent Wave Theory for Guided Modes in Graded Index Optical Fibers

J. M. ARNOLD AND LEOPOLD B. FELSEN, FELLOW, IEEE

*Abstract*—**The evanescent wave theory of modal propagation in graded index optical fibers, developed recently by Choudhary and Felsen, is based on certain postulates, which are examined here. It is shown that an analyticity condition on the asymptotic expansion coefficients for the modal amplitudes on the fiber axis, as imposed by these authors, should be replaced by the single-valuedness of these coefficients in a strip of the complex coordinate plane. The analyticity condition is of questionable validity because of the nonuniformity of the asymptotic expansion on the axis. The previous *results* of Choudhary and Felsen are found to be unaffected by this change but the method is now made rigorous and need not be justified, as before, by comparison with asymptotic expansions of exact solutions for special profiles. Also developed here is a uniform asymptotic approximation that is valid near the fiber axis and connects with the leading term of the nonuniform evanescent wave theory formulation. Within this rigorous framework, the evanescent wave theory continues to provide a very useful and systematic procedure for calculating modal eigenvalues and modal fields to arbitrary orders in inverse powers of the large wavenumber $k$.**

## I. Introduction and Summary

THE EVANESCENT wave theory of Choudhary and Felsen [1]–[3], developed as an extension of geometrical optics to permit the tracking of local plane wave fields with complex phase, has been applied by these authors to guided propagation in slab [4] and cylindrical [5] waveguides with a refractive index profile represented by an analytic function of the transverse coordinates. To obtain sufficient conditions for the determination of the coefficients in the asymptotic expansions of the modal eigenvalues and amplitudes in inverse powers of the large wavenumber $k$, Choudhary and Felsen utilized an analyticity property satisfied by the true modal field on the waveguide axis. However, since their asymptotic expansion is nonuniform near the axis, the imposition of analyticity on the expansion coefficients is questionable. The procedure was nevertheless rendered plausible since the results agree term by term with those of available exact solutions in slab or cylindrical configurations, when these special cases were considered.

In a separate publication [6], the evanescent wave theory for slab waveguides was reexamined. It was shown that the asymptotic modal amplitude coefficients must be single valued in a strip of the complex transverse coordinate plane, but that no statement of analyticity on the waveguide axis (the transverse coordinate origin) need be made. This new condition turns out to have precisely the same effect as that of analyticity, but the method is now rigorous and need not be justified by comparison with special exact solutions. Derived as well was a uniform asymptotic representation, which permits construction of the leading asymptotic term of the nonuniform expansion, and also of a local parabolic approximation near the waveguide axis, as advocated by Choudhary and Felsen to cope with the convergence problem of their expansion near the origin. So formulated, the evanescent wave theory continues to furnish a mechanism for systematic and tractable determination of the asymptotic expansions of the modal eigenvalues and eigenfunctions to any desired order.

In the present paper, the method introduced in [6] for the slab waveguide is applied to the fiber geometry. Since the procedure for the fiber follows closely that of the slab, we present the results and conclusions in summary form. For details the reader is referred to the above-mentioned paper [6].

## II. The Asymptotic Expansion

We consider a radially stratified medium with refractive index $n(r)$

$$n^2(r)=n_0^2\{1-f^2(r)\}, \qquad 0<r<\infty \tag{1}$$

where $n_0=n(0)$ and $f(r)$ has the polynomial form

$$f=a_0 r\prod_{j=1}^{M}\left(1+b_j r^2\right), \qquad a_0>0,\, b_j>0. \tag{2}$$

The real constants $\{b_j\}$ must be positive since we require that the only real zero of $f$ is at the origin on the real axis [6], and to ensure proper behavior of the transverse modal field solutions at infinity. The differential equation for the radial portion $\phi$ of the transverse modal field is

$$r\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)-\left\{\left(n_0^2k^2f^2-\gamma^2\right)r^2-m^2\right\}\phi=0 \tag{3}$$

with

$$\gamma^2=n_0^2k^2-\beta^2 \tag{4}$$

and $\beta$ representing the propagation coefficient along the axial coordinate $z$. The integer $m$ is the azimuthal mode

Manuscript received October 26, 1979; revised April 30, 1980. The work of J. M. Arnold was supported by the Post Office Research Centre, Martlesham, Suffolk, England, and the work of L. B. Felsen was supported by the Joint Services Electronics Program under Contract F496-20-78C0074.

J. M. Arnold is with the Department of Electrical and Electronic Engineering, University of Nottingham, Nottingham, England.

L. B. Felsen is with the Department of Electrical Engineering, Polytechnic Institute of New York, Farmingdale, NY 11735.

0018-9480/80/0900-0996$00.75 ©1980 IEEE

number. Thus, the modal field $u(r,\theta,z)$ is expressed as

$$u=\phi(r)\exp(im\theta)\exp(i\beta z). \qquad (5)$$

Note that, since we shall be looking for solutions with $\beta \sim n_0 k$ as $k \to \infty$, the coefficient $\gamma^2$ is $0(k)$ in (3). Moreover, for the profile satisfying (2), $\phi$ must be a single valued function of $r$. By treating $r$ as a complex variable and performing certain contour integrations, this condition is used subsequently for determination of the asymptotic expansion coefficients of the propagation coefficient. It then follows that linear terms of the form $(a_j r)$ in the cylindrical profile function (2) are not admitted since the profile must remain unchanged for negative values of $r$. Such a restriction does not arise in the slab geometry [6].

By letting

$$R=\phi^{-1} r \frac{d\phi}{dr} \qquad (6a)$$

$$\phi=\exp\left\{\int \frac{R}{r} dr\right\} \qquad (6b)$$

one obtains the Riccati equation

$$r\frac{dR}{dr}+R^2=(n_0^2k^2f^2-\gamma^2)r^2-m^2 . \qquad (7)$$

As in [6] we employ the asymptotic expansions

$$(n_0k)^{-1}\gamma^2=\chi \sim \sum_{p=0}^{\infty}(n_0k)^{-p}\chi_p \qquad (8a)$$

and

$$R \sim \sum_{p=0}^{\infty}(n_0k)^{-p+1}R_p . \qquad (8b)$$

Thus, from (7), equating like powers of $k$,

$$R_0=(f^2r^2)^{1/2}=-fr \qquad (9)$$

in order to ensure vanishing of $\phi$ at infinity. The higher order coefficients are obtained recursively from the equations

$$-2R_0R_1=\chi_0 r^2+r\frac{dR_0}{dr} \qquad (10a)$$

$$-2R_0R_2=\chi_1 r^2-m^2+r\frac{dR_1}{dr}+R_1^2 \qquad (10b)$$

$$-2R_0R_p=\chi_{p-1}r^2+r\frac{dR_{p-1}}{dr}+\sum_{s=1}^{p-1}R_sR_{p-s}, \qquad p\geq 2 \qquad (10c)$$

provided that the eigenvalue coefficients $\{\chi_p\}$ are known.

## III. Mode Quantization

To determine the $\{\chi_p\}$ we use contour integration as in [6] but now in the complex $r$ plane (see also [7]). Since $\phi$ must be single valued, we must have for arbitrary $r$

$$\phi(r)=\phi(re^{2\pi i})=\phi(r)\exp\{\oint_c r^{-1}R dr\} \qquad (11)$$

(by (6b)) and therefore

$$\oint_c r^{-1}R dr=2N\pi i \qquad (12)$$

where $N$ is an integer. By choosing $c$ (which is an arbitrary closed contour) such that it encloses a strip surrounding the real axis, and applying the residue theorem to (12) with (6a), it is possible to identify $N$ with the number of zeros of $\phi$ on the real axis. Hence

$$\oint_c r^{-1}R dr=2\pi i(2q+m) \qquad (13)$$

where $q$ is a nonnegative integer, and the integer $m$ appears because $\phi$ has a zero of multiplicity $m$ at $r=0$. Since, from (8)–(10), the asymptotic expansion (8b) is uniform everywhere except at the zeros of $f$, $R$ may be replaced by this expansion under the integral sign in (13) provided that $c$ does not pass through any zeros of $f$. Hence, equating each asymptotic order separately in (13), we obtain

$$\oint_c r^{-1}R_0 dr=0 \qquad (14a)$$

$$\oint_c r^{-1}R_1 dr=2\pi i(2q+m) \qquad (14b)$$

$$\oint_c r^{-1}R_p dr=0, \qquad p\geq 2. \qquad (14c)$$

Equation (14a) is satisfied because of (9) and the assumed analyticity of $f$. The contour integrals in (14) represent all the conditions necessary to determine the coefficients $\{\chi_p\}$. From (10a)

$$r^{-1}R_1=\tfrac{1}{2}f^{-1}\left\{\chi_0-r^{-1}\frac{d}{dr}(fr)\right\} \qquad (15)$$

whence from (14b) and (2)

$$\chi_0=2(2q+m+1)a_0. \qquad (16)$$

It also follows from (9) and (10) that $R_p$ has the following form:

$$r^{-1}R_p=F_p(r)+\sum_{s=0}^{p-1}c_s^{(p)}r^{-(2s+1)} \qquad (17)$$

where $F_p(r)$ is an analytic function of $r$. This form arises from the combined effect of the operations performed in (10) to obtain $R_p$ from earlier terms in the sequence: differentiation of $R_{p-1}$, which has a pole of order $2p-1$ at $r=0$, and then division by $f$, which has a zero at $r=0$. Thus, the singularity in $R_p$ is two orders higher than that of $R_{p-1}$. Equations (17) and (14) imply that the simple pole coefficient must vanish, where

$$c_0^{(p)}=0. \qquad (18)$$

Since

$$\int r^{-1}R_p dr=\int F_p(r)dr-\sum_{s=1}^{p-1}\frac{c_s^{(p)}r^{-2s}}{2s}+c_0^{(p)}\ln r \qquad (19)$$

condition (18) eliminates all logarithmic terms from the asymptotic expansion of $\phi$ in (6b) that results when the exponential factors are expanded in inverse powers of $k$. This prescription for determining the $\{\chi_p\}$ coefficients was previously used by Choudhary and Felsen [5], but on the questionable grounds of analyticity of the asymptotic expansion at the origin, instead of the more rigorous single-valuedness argument employed here. The results of Choudhary and Felsen therefore remain valid, and the evanescent wave theory guarantees provision of the rigorous asymptotic expansion coefficients in (8a) and (8b) for the class of profiles covered by (2). Note that while the expansion in (8b) for the modal amplitudes becomes invalid as $r \to 0$, the expansion in (8a) for the modal eigenvalues is unaffected by that restriction. A numerical algorithm for the determination of the expansion coefficients has been developed [8] and permits the calculation of the eigenvalues to a high degree of accuracy.

## IV. Uniform Representation

Uniform asymptotic expansions, unlike the nonuniform results of the evanescent wave theory, provide access to the field on and near the fiber axis. Such expansions for cylindrical fiber waveguides have been investigated elsewhere [9], [10], and here we shall only summarize the results. As in [6], the Liouville transformation from $r$ to $\xi$,

$$(\xi^2-\xi_0^2)\left(\frac{d\xi}{dr}\right)^2=f^2-f_0^2, \qquad f_0^2=(n_0k)^{-1}\chi \tag{20}$$

together with

$$\phi=\left(r\frac{d\xi}{dr}\right)^{-1/2}\Phi \tag{21}$$

transforms (3) into

$$\frac{d^2\Phi}{d\xi^2}+\left\{n_0^2k^2(\xi_0^2-\xi^2)+\frac{\mu}{\xi^2}+h\right\}\Phi=0, \qquad \mu=\frac{1}{4}-m^2 \tag{22}$$

where $h$ is an $0(1)$ function in $k$. If it remains $0(1)$ uniformly on some part of the real $r$ axis, then it can there be neglected as $k\to\infty$ in (22), leading to the approximation

$$\phi\sim\left(r\frac{d\xi}{dr}\right)^{-1/2}\Phi_0 \tag{23}$$

where

$$\frac{d^2\Phi_0}{d\xi^2}+\left\{n_0^2k^2(\xi_0^2-\xi^2)+\frac{\mu}{\xi^2}\right\}\Phi_0=0. \tag{24}$$

By integrating (20), $\xi$ is implicitly given by

$$\int_{\xi_0}^{\xi}(\xi'^2-\xi_0^2)^{1/2}d\xi'=\int_{r_0}^{r}\left[f^2(r')-f_0^2\right]^{1/2}dr' \tag{25}$$

$r_0$ being the value of $r$ for which $f=f_0$. Analyticity of the mapping is ensured by having $\xi_0$ correspond to $r_0$ and $\xi=0$ to $r=0$ when

$$\frac{\pi}{4}\xi_0^2=\int_0^{r_0}(f_0^2-f^2)^{1/2}dr. \tag{26}$$

Solutions of (24) are related to the Whitaker functions. In view of the boundary conditions at $r=0$ and $r\to\infty$ relevant here, they are known to reduce to the Laguerre polynomial [11], [12]

$$\Phi_0=\xi^{m+1/2}L_q^{(m)}\{(n_0k)\xi^2\}\exp\{-n_0k\xi^2/2\} \tag{27}$$

where

$$2(2q+m+1)=n_0k\xi_0^2. \tag{28}$$

The two conditions (26) and (27) determine $\chi$, and it can be proved that $\chi$ found in this manner agrees to within its first asymptotic order with its value from Section III, similar to the planar waveguide calculation in [6]. Higher order terms cannot be checked directly, because they are implicitly neglected in the uniform approximation (23). This is an unsatisfactory feature of uniform approximations which finds a simple remedy in the Choudhary and Felsen evanescent wave method. Here [5] it is quite easy to get higher order terms for the eigenvalue whereas improvement of (23) is rather involved.

As in [6], (27) can be used to obtain approximate expressions for both large *and small* $r$. By approximating $\xi$ in the form

$$\xi(r)=\eta_0(r)+(n_0k)^{-1}\eta_1(r)+0(k^{-2}) \tag{29}$$

and evaluating $\eta_0$ and $\eta_1$ by using (29) in (20), it can be shown that as $k\to\infty(r\neq0)$, $\phi$ can be approximated by

$$\phi\sim C_1(rf)^{-1/2}r^{2q+m+1}\exp\left\{-n_0k\int_0^r fdr'\right\}\cdot\exp\left\{(2q+m+1)\int_0^r\left(\frac{a_0}{f}-\frac{1}{r'}\right)dr'\right\} \tag{30a}$$

where

$$C_1=\frac{(-1)^q}{q!}(n_0k)^q. \tag{30b}$$

Equation (30) is to be compared with [5], from which it will be found that the evanescent wave approximation is

$$\phi\sim C(rf)^{-1/2}\exp\left\{-n_0k\int^r fdr'\right\}\cdot\exp\left\{(2q+m+1)\int^r\frac{a_0dr'}{f}\right\} \tag{31}$$

where $C$ is arbitrary. These two expressions differ only in the fixed lower limits of zero in (30a), with some rearrangement to ensure that the integrals exist at this limit.

On the other hand, $r$ can be permitted to go to zero in the uniform approximation, which cannot be allowed in the evanescent wave approximation as this approximation is nonuniform at $r=0$. Thus, letting $r^2\sim0(k^{-1})$, evaluating the appropriate expression (20) for $\xi$ as $k\to\infty$, and substituting in (27), leads to

$$\phi\sim C_2r^m\exp(-n_0ka_0r^2/2)L_q^{(m)}(n_0ka_0r^2) \tag{32a}$$

where

$$C_2=a_0^{m/2}. \tag{32b}$$

The approximation in (32) is uniform at the *single point* $r=0$ as $k\to\infty$, but will be accurate for finite large $k$ in a small neighborhood of the origin. A uniformization procedure for the evanescent wave theory, valid at $r\to0$ and improving (32a), could be attempted by extracting in closed form the exact solution for the parabolic profile (see [4]). This procedure, yet to be tested, may provide adequate numerical overlap with the region covered by the nonuniform theory.

A final comment concerning the profile form in (2) is in order. Choudhary and Felsen [5] considered also the case where a linear term of the form $a_1r$ is included. It was pointed out by Ramskov-Hansen and Jacobsen [8] that the asymptotic expansion of the modal propagation coefficient is insensitive to the algebraic sign of $a_1$ (see [5]), although positive and negative values of $a_1$ define different refractive index profiles. The discussion in Section II shows that linear terms cannot be accommodated and therefore this class of profiles must be excluded from those that can be analyzed by the evanescent wave method. The same behavior with respect to $a_1$ occurs for the slab waveguide but here the different profiles corresponding to positive and negative $a_1$ are merely reflections of one another about the waveguide axis and therefore have the same modal propagation coefficient. Thus, as shown in [6], the asymmetric slab waveguide is included within the framework of the evanescent wave theory.

## REFERENCES

[1] S. Choudhary and L. B. Felsen, "Asymptotic theory for inhomogeneous waves," *IEEE Trans. Antennas Propagat.*, vol. AP-21, pp. 827–842, 1973.

[2] ——, "Analysis of Gaussian beam propagation and diffraction by inhomogeneous wave tracking," *Proc. IEEE*, vol. 62, pp. 1530–1541, 1974.

[3] L. B. Felsen, "Evanescent waves," *J. Opt. Soc. Amer.*, vol. 66, pp. 751–760, 1976.

[4] S. Choudhary and L. B. Felsen, Asymptotic theory of ducted propagation," *J. Acoust. Soc. Amer.*, vol. 63, pp. 661–666, 1978.

[5] ——, "Guided modes in graded index optical fibers," *J. Opt. Soc. Amer.*, vol. 67, pp. 1192–1196, 1977.

[6] J. M. Arnold and L. B. Felsen, "Rigorous asymptotic theory of evanescent waves for guided propagation," *J. Acoust. Soc. Amer.*, vol. 67, pp. 757–763, 1980.

[7] J. L. Dunham, "The Wentzel-Brillouin-Kramers method of solving the wave equation," *Phys. Rev.*, vol. 41, pp. 713–720, 1932.

[8] J. J. Ramskov-Hansen and G. Jacobsen, private communication.

[9] J. M. Arnold, "Inhomogeneous dielectric waveguides: A uniform asymptotic theory," *J. Phys.*, to be published.

[10] ——, "Uniform asymptotic theory of inhomogeneous waveguide modes near cut-off," *J. Phys.*, to be published.

[11] L. J. Slater, *Confluent Hypergeometric Functions*. Cambridge, England: Cambridge University Press, 1960.

[12] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions and Tables*. New York: Dover, 1965.

# Excitation of Surface Waves and the Scattered Radiation Fields by Rough Surfaces of Arbitrary Slope

EZEKIEL BAHAR, SENIOR MEMBER, IEEE

***Abstract*—Surface waves as well as lateral waves are excited when a rough surface is illuminated by the radiation fields. In view of shadowing, these terms of the complete field expansions contribute significantly to the total fields when the transmitter or receiver are near the rough surface. In this work explicit expressions are derived for the coupling between the radiation fields and the surface waves which are guided at the irregular interface between two media. In the analysis, the slope of the rough surface is not restricted and the solutions for both the horizontally and vertically polarized waves are shown to satisfy reciprocity and duality relationships in electromagnetic theory. Special consideration is given to Brewster angles of incidence and scatter and stationary phase techniques. The full-wave solutions are also applied to random and periodic rough surfaces.**

## I. INTRODUCTION

USING A full-wave approach that accounts for shadowing, it has been shown that the radiation fields scattered from rough surfaces vanish in a continuous manner as the observer moves into the shadow region [4]. Thus when the transmitter or receiver are near the rough boundary, the major contributions to the total fields come from the surface wave and the lateral wave terms of the complete field expansions [1], [2].

In this paper the full-wave approach is used to determine the excitation of the surface wave when the rough surface is illuminated by the radiation field. In addition the scattered radiation fields excited by an incident surface wave are determined. The Kirchoff approach or the Rayleigh hypothesis for instance, cannot be used to solve this problem [3], [6]. Both vertically and horizontally polarized waves are considered and the solutions are shown to satisfy duality and reciprocity relationships in electromagnetic theory.

For the convenience of the reader, the principal elements of the full-wave approach, including the complete expansions of the fields, the exact boundary conditions and the rigorous set of coupled differential equations for the wave amplitudes (generalized telegraphist's equations) are summarized in Section II. In addition explicit expressions for the coupling coefficients are provided.

In Section III second-order iterative solutions for the scattered fields are presented. To remove the small slope restriction inherent in the iterative solutions (while at the same time retaining the relatively simple form of these

Manuscript received January 22, 1980; revised April 30, 1980.

The author is with the Electrical Engineering Department, University of Nebraska, Lincoln, Nebraska 68588.